$f(R)$ Gravity and Chameleon Theories

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We analyse $f(R)$ modifications of Einstein's gravity as dark energy models in the light of their connection with chameleon theories. Formulated as scalar-tensor theories, the $f(R)$ theories imply the existence of a strong coupling of the scalar field to matter. This would violate all experimental gravitational tests on deviations from Newton's law. Fortunately, the existence of a matter dependent mass and a thin shell effect allows one to alleviate these constraints. The thin shell condition also implies strong restrictions on the cosmological dynamics of the $f(R)$ theories. As a consequence, we find that the equation of state of dark energy is constrained to be extremely close to -1 in the recent past. We also examine the potential effects of $f(R)$ theories in the context of the Eöt-wash experiments. We show that the requirement of a thin shell for the test bodies is not enough to guarantee a null result on deviations from Newton's law. As long as dark energy accounts for a sizeable fraction of the total energy density of the Universe, the constraints which we deduce also forbid any measurable deviation of the dark energy equation of state from -1. All in all, we find that both cosmological and laboratory tests imply that $f(R)$ models are almost coincident with a ΛCDM model at the background level.

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I. INTRODUCTION

The acceleration of the Universe expansion was discovered ten years ago and is still a deep mystery (see e.g. [1] for recent results on observations of dark energy and e.g. [2, 3] for theoretical overviews). Two types of approaches have been considered. One can either introduce a new kind of matter whose role is to trigger acceleration or modify the behaviour of gravity at cosmological distances. In the first approach, dark energy is a new energy form, with all its well-known puzzles such as the cosmological constant problem, the coincidence problem and the value of the equation of state. This approach is subject of intense experimental investigation and any deviation from -1 would be a smoking gun for new physics beyond the standard models of particle physics and cosmology. On the other hand, in the second approach, various attempts to modify gravity have been presented (see e.g. [4]-[14]; the literature is vast, see [15] for a recent overview and further references). Up to now, they are all plagued with various theoretical problems such as the existence of ghosts or instabilities. In this paper, we will consider a modification of Einstein's gravity, the so–called $f(R)$ theories, which do not seem to introduce any new type of matter and can lead to late time acceleration. In fact, these theories can be reformulated in terms of scalar-tensor theories with a fixed coupling of the extra scalar degree of freedom to matter. As theories of dark energy, they suffer from the usual problems and are also potentially ruled out by gravitational tests of Newton's law.

The only way-out for these models is to behave as chameleon theories [16], i.e. develop an environment dependent mass [17, 18, 19, 20]. When the density of the ambient matter in which the scalar field/chameleon propagates is large enough, its mass becomes large and the smallness of the generated fifth force range is below the detectability level of gravitational experiments. On the other hand, for planetary orbits or any other situations in which gravity is at play in a sparse environment, one must impose the existence of the so–called thin shell effect. In this case, the fifth force is attenuated as the chameleon is trapped inside very massive bodies (the Sun for instance). It has been argued that the existence of thin shells is usually enough to salvage $f(R)$ models [17, 19]. We show that thin shells do not always guarantee null results in experimental tests of Newton's law. We exemplify this fact using the Eöt-wash setting and obtain strong constraints on the models which translate into stringent bounds on the present dark energy equation of state, preventing any detection of a deviation from -1 in the foreseeable future $|1+w| \le 10^{-4}$, where w is the equation of state of dark energy in the recent past. This corroborates a similar bound obtained from the existence of thin shell for objects embedded in a super-cluster. It should be noted that this result holds at the background level. For higher red-shifts where the effective dark energy density fraction, $\Omega_{\rm de}$, may become small (or even vanish), larger deviations can be present as exemplified in the models in [21, 22] where the equation of state can deviate from -1 for red-shifts of order $z = 2 - 3$. In all these models however $|1+w|\Omega_{\rm de} \ll 1$, and so even if w deviates significantly from -1, deviations of the homogeneous cosmology from ΛCDM are still very small. Detectable deviations from ΛCDM are envisageable at the perturbative level as the growth factor is anomalous at small scales (see e.g. [23] for a discussion of this point for the original chameleon model). Some consequences of this fact on the matter power spectrum and the CMB spectrum of $f(R)$ models have been presented in Ref. [24, 25, 26].

The paper is organized as follows: In the subsequent section, we review $f(R)$ models and chameleon theories. In section III we derive the cosmological thin shell bound on the equation of state. In section IV, we consider tests of the inverse square law. Finally, we apply these considerations to particular models in section V. The appendices contain some technical details.

II. $f(R)$ GRAVITIES AND CHAMELEON THEORIES

A. $f(R)$ theories

An $f(R)$ theory is a modified gravity theory in which the usual Einstein-Hilbert Lagrangian for General Relativity, i.e. R, is replaced by some arbitrary function of the scalar curvature i.e. $f(R)$. The action for an $f(R)$ gravity theory therefore takes the following form:

$$S_{f(R)} = \int \mathrm{d}^4 x \sqrt{-g} \frac{M_{\rm Pl}^2}{2} f(R) + S_{\rm matter}[g_{\mu\nu}, \Psi_i], \quad (1)$$

where the Ψ_i represent the matter fields.

In this work we are concerned only with metric $f(R)$ theories, in which only the metric $g_{\mu\nu}$ is the independent variable in the gravitational sector. The quantity $\Gamma^{\rho}_{\mu\nu}$ is taken to be the Levi-Civita connection associated with the metric $g_{\mu\nu}$. In these metric $f(R)$ theories the field equations are:

$$\begin{aligned} R_{\mu\nu} f'(R) & - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T^{\rm matter}_{\mu\nu} \\ & + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R). \end{aligned} \quad (2)$$

where $\kappa = 1/M_{\rm Pl}^2$.

B. Transformation to a Scalar-Tensor theory

Eq. (2) gives a set of equations which are second order in derivatives of R, which is itself second order in derivatives of $g_{\mu\nu}$, making the field equations fourth order in $g_{\mu\nu}$. Finding solutions to fourth order equations can be mathematically and physically troublesome, but fortunately metric $f(R)$ theories can be recast as a scalar tensor theory with only second order equations via a well known conformal transformation. We define ϕ by

$$\exp\left(-\frac{2\beta\phi}{M_{\rm Pl}}\right) = f'(R),$$

where $\beta = \sqrt{1/6}$. We also define the *Einstein frame* metric $\bar{g}_{\mu\nu}$ by a conformal transformation

$$\bar{g}_{\mu\nu} = e^{-\frac{2\beta\phi}{M_{\rm Pl}}} g_{\mu\nu},$$

and let $\bar{R}$ be the scalar curvature of $\bar{g}_{\mu\nu}$. When rewritten in terms of $\bar{g}_{\mu\nu}$ and ϕ, Eq. (1) becomes:

$$\begin{aligned} S_{\rm ST} = & \int \mathrm{d}^4 x \sqrt{-\bar{g}} \left(\frac{M_{\rm Pl}^2}{2} \bar{R} - \frac{1}{2} \bar{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \\ & + S_{\rm matter}[e^{\frac{2\beta\phi}{M_{\rm Pl}}} \bar{g}_{\mu\nu}, \Psi_i], \end{aligned} \quad (3)$$

where the potential $V(\phi)$ is given by:

$$V(\phi) = \frac{M_{\rm Pl}^2 \left(R f'(R) - f(R)\right)}{2 f'(R)^2}.. \quad (4)$$

When the action is written in the form of Eq. (3), we say that we are working in the Einstein frame. The field equations then become:

$$\begin{aligned} \bar{G}_{\mu\nu} &= \bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} \bar{g}_{\mu\nu} = \kappa \bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi \quad (5) \\ & \quad - \kappa \bar{g}_{\mu\nu} \left[\frac{1}{2} (\bar{\nabla}\phi)^2 + V(\phi) \right] + \kappa T^{\rm matter}_{\mu\nu} \\ \bar{\Box}\phi &= V'(\phi) - \frac{\beta}{M_{\rm Pl}} T^{\rm matter}. \quad (6) \end{aligned}$$

In the above and subsequent expressions, the covariant derivatives, $\bar{\nabla}_\mu$, obey $\bar{\nabla}_\mu \bar{g}_{\mu\nu=0}$ and all indices are raised and lowered with $\bar{g}_{\mu\nu}$ unless stated otherwise. We note that in the Einstein frame $T^{\rm matter}_{\mu\nu}$ is not conserved but instead:

$$\bar{\nabla}_\mu T^{\rm matter\,\mu}{}_\nu = \frac{\beta}{M_{\rm Pl}} T^{\rm matter} \bar{\nabla}_\nu \phi. \quad (7)$$

This implies that matter will generally feel a new or 'fifth' force due to gradients in ϕ. We note from Eq. (5) that, when written as a scalar tensor theory, gravity in an $f(R)$ theory is essentially General Relativity, and all the modifications are essentially due to the effective 'fifth force' and to the energy density of ϕ. Much of our intuition for how gravity works is based on how it works in General Relativity. When an $f(R)$ theory is written as a scalar tensor theory we can readily make use of this intuition in solving the field equations. This may not be the case, however, in the original frame in which the equations were fourth order and so in those circumstances one would have to be more careful. Note that all physical observables must be independent of the choice of frame, i.e. the choice of metric $g_{\mu\nu}$ or $\bar{g}_{\mu\nu}$.

C. Chameleon Theories

Since $f(R)$ theories are equivalent to scalar tensor theories one can generally directly apply the plethora of constraints on scalar tensor models. In particular, since the extra degree of freedom, ϕ, couples to matter with gravitational strength, tests of the inverse square law such as the Eöt-Wash experiment require that ϕ have a mass, $m_\phi = \sqrt{V_{,\phi\phi}}$, greater than 1 meV. Cosmologically ϕ would then have been fixed at its minimum since very early times, and physics over astrophysical scales would be indistinguishable from predicted by unmodified General Relativity with a cosmological constant. Both the coincidence problem and the problem of the small size of the cosmological constant would not be alleviated in this scenario. However, this is not the whole story. Laboratory constraints on scalar tensor theories can be greatly relaxed if $m_\phi = \sqrt{V_{,\phi\phi}}$ develops a strong dependence on the ambient density of matter. Theories in which such a dependence is realized are said to have a chameleon mechanism and to be chameleon theories. In such theories, ϕ can be heavy enough in the environment of the laboratory tests so as to evade them, whilst remaining relatively light on cosmological scales. It must be stressed that even with a chameleon mechanism, it is still very difficult, if not impossible, to construct such a theory where the late time cosmology would be observational distinguishable from the usual ΛCDM model. To the best of our knowledge all such theories which are also experimentally viable require a fairly high degree of fine tuning to ensure that the effective cosmological constant is small enough.

A chameleon theory is essentially just a scalar-tensor theory in which the potential has certain properties. As such Eqs. (3 - 7) also define a chameleon theory for certain classes of $V(\phi)$. In these circumstances the $f(R)$ theory would be equivalent to a chameleon theory. In a general chameleon theory, β, which parametrizes the strength of the coupling of ϕ to matter, could take any value and potentially even be different for different matter species. If a chameleon theory is equivalent to a $f(R)$ theory, however, β is fixed to be $\sqrt{1/6}$ and is the same for all types of matter. If a $f(R)$ theory is not equivalent to a chameleon theory it would be generally ruled out by laboratory tests of gravity and / or result in no detectable deviations from General Relativity over astrophysical scales.

For an $f(R)$ theory to have a chameleon mechanism one must require that, in at least some region of ϕ-space:

$$V'(\phi) < 0, \qquad V''(\phi) > 0, \qquad V'''(\phi) < 0.$$

It follows from the definition of ϕ that:

$$\frac{\mathrm{d}\phi}{\mathrm{d}R} = -\frac{M_{\mathrm{Pl}}}{2\beta}\frac{f''}{f'}. \tag{8}$$

and therefore the derivatives follow

$$V'(\phi) = \frac{\beta M_{\mathrm{Pl}}}{f'^2}\left[Rf' - 2f\right], \tag{9}$$

$$V''(\phi) = \frac{1}{3}\left[\frac{R}{f'} + \frac{1}{f''} - \frac{4f}{f'^2}\right], \tag{10}$$

$$V'''(\phi) = \frac{2\beta}{3M_{\mathrm{Pl}}}\left[\frac{3}{f''} + \frac{f'f'''}{f''^3} + \frac{R}{f'} - \frac{8f}{f'^2}\right] \tag{11}$$

In general, this gives strong constraints on the form of $f(R)$. In the following we will study examples where these conditions are met.

When these conditions are satisfied, the mass of ϕ in a suitable large region with density ρ will increase with ρ. In order to evade constraints coming from local tests of gravity, it is not, however, enough that a theory possess a chameleon mechanism; the mechanism must, in addition be strong enough for chameleonic behaviour to occur for the test masses used in the laboratory gravity experiments.

D. Thin-Shells

1. *Chameleon Theories*

Chameleon theories do not behave like linear theories of massive scalar fields. In situations where massive bodies are involved, the chameleon field is trapped inside such bodies and its influence on other bodies is only due to a thin shell at the outer edge of a massive body[16]. As a result, the field outside the massive body for distances less than the range of the chameleon force in the outer vacuum is effectively damped leading to a shielded fifth force which becomes undetectable. The criterion for a thin shell is

$$\frac{\Delta\phi}{m_{\mathrm{Pl}}} \leq 2\beta\Phi_N \tag{12}$$

where $\Delta\phi = \phi_\infty - \phi_0$ is the field difference from far inside the body to very far away. We define the body and the region outside it to have densities ρ_0 and ρ_∞ respectively. It involves Newton's potential Φ_N at the surface of the body. In general the field values at infinity, ϕ_∞, and deep inside, ϕ_0, are related to ρ_∞ and ρ_0 by

$$\partial_\phi V = -\beta\frac{\rho}{m_{\mathrm{Pl}}}. \tag{13}$$

In most current situations involving runaway potentials, when $\rho_0 \gg \rho_\infty$, this implies that $\phi_\infty \geq \phi_0$. Hence, $\Delta\phi = \phi_\infty$ implying that cosmological information can be inferred from local tests. Moreover, in a cosmological setting, the chameleon sits at the minimum (13) during the matter era. As a result, the variation of the equation of state in the recent past is severely constrained. Another important consequence of the chameleon effect is the existence of an anomalous growth of the density contrast for scales lower than the inverse mass of the chameleon, i.e. it grows like a^ν where $\nu \approx \frac{-1+\sqrt{1+24(1+2\beta^2)}}{2}$[23]. In the $f(R)$ setting, some of the consequences of this anomalous growth on the CMB and the matter power spectrum have been analysed using the convenient variable

$$B = \frac{f_{RR}}{f_R}\frac{dR}{dH}H \tag{14}$$

whose square root represents the compton wave-length, i.e. the inverse mass of the chameleon, in horizon units

H^{-1}[24, 25, 26]. Effects on structure formation could be seen for values as low as $B = 10^{-4}$ in future galaxy surveys[25]. In the following we will find an explicit example of logarithmic $f(R)$ model which could lead to effects on scales as large as 100 h^{-1}Mpc. All these facts will be crucial in the following.

2. Thin shells in the language of $f(R)$ theories

It is useful to write the function $f(R)$ in the form $f(R) = R + h(R)$, where h measures the deviation to Einstein gravity. To leading order, as a consequence of Ref. [16], the thin shell condition can be formulated as [26]

$$|\Delta h'(R)| \leq \frac{2}{3}\Phi_N. \tag{15}$$

As Newton's potential is small on cosmological scales, with an upper bound around 10^{-4}, this implies that h' must have very small variations. The thin shell condition is a constraint on local experiments at the present time. It has nothing to say, a priori, about the evolution of the universe since matter equality for instance. Another useful combination (which is not to be confused with the chameleon mass m_ϕ) has been used

$$m = \frac{Rh''(R)}{1 + h'(R)} \tag{16}$$

It has been shown that the existence of a matter era followed by an accelerated period requires $m < 0.1$. For models where m is (nearly) a power law, the thin shell constrain implies that m is much smaller for reasonable powers. In the following, we will obtain a bound on the equation of state at present time which implies that departures from ΛCDM are tiny.

III. THIN-SHELL CONSTRAINTS ON COSMOLOGY

In subsequent sections we will assume that test bodies used in laboratory based gravity experiments have thin-shells. In the absence of any thin-shell, the inverse square law tests, such as the Eöt-Wash experiment [32] (as well as other tests of gravity over longer ranges), rule out theories with $\beta = 1/\sqrt{6}$ as it is in $f(R)$ theories. The thin-shell requirement must therefore be satisfied by any physically viable $f(R)$ theory. Although it is not often appreciated, the thin-shell condition for laboratory test masses actually places extremely tight constraints on the recent cosmological evolution of ϕ. In this section we consider those constraints in the context of a general $f(R)$ theory.

In any single field scalar tensor theory there is a choice of frame. In the Jordan frame, the laws of physics in a local inertial frame are the same everywhere, however Newton's constant, G_N, is different at different points in space and time. In the Einstein frame, G_N is chosen to be fixed but, as a result, local particle physics is position dependent. The process of converting astronomical observations to cosmological parameters generally involves making assumptions about how today's laws of particle physics are related to those in the past. This said, if the relative changes in G_N (in the Jordan frame) are small i.e. $\ll 1$, the differences between cosmological parameters in the two frames are only very slight. For instance, to calculate a redshift, one must compare the observed wavelength, $\lambda_{\rm obs}$ of a particular absorption or emission band to the wavelength that band would have had at emission, $\lambda_{\rm e}$. Since one cannot go to the astronomical object in question and directly observe the wavelength at emission, it is generally assumed that particles physics in the past obeyed the same laws as it does today and so replace $\lambda_{\rm e}$ with the wavelength of the band as it is measured in a laboratory today, $\lambda_{\rm today}$. When one is dealing with scalar-tensor theories, the assumption that $\lambda_{\rm e} = \lambda_{\rm today}$ is equivalent to a choice of frame, specifically the Jordan frame.

To make comparison with observations straightforward, one should therefore quote cosmological parameters for the Jordan frame. This said, it is often more straightforward to perform calculations in the Einstein frame and then merely express the results in terms of Jordan frame quantities.

Cosmologically, in the Jordan frame we have:

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right], \tag{17}$$

and ϕ obeys:

$$-\frac{1}{a^2}\Phi_{,\eta\eta} - 2\frac{a_{,\eta}}{a^3}\Phi_{,\eta} = \frac{\kappa}{3}\left[T_{\rm matter} + 2\Phi^3 V_{,\Phi}\right] \tag{18}$$

where $\Phi = e^{-2\beta\phi/M_{\rm Pl}} = f'(R)$. At late times, when it is appropriate to ignore the contribution of radiation to the total energy density of the Universe, we have

$$\frac{3a_{,\eta}^2}{a^4} = \frac{\kappa\rho_{\rm matter}}{\Phi} + \kappa\Phi V(\phi) - \frac{3a_{,\eta}\Phi_{,\eta}}{a^3\Phi} - \frac{3k}{a^2}. \tag{19}$$

The Einstein equations also give:

$$\frac{2a_{,\eta\eta}}{a^3} - \frac{a_{,\eta}^2}{a^4} = \kappa\Phi V(\phi) - \frac{\Phi_{,\eta\eta}}{\Phi a^2} + \frac{2k}{a^2} - \frac{a_{,\eta}\Phi_{,\eta}}{a^3\Phi}. \tag{20}$$

We assume that measurements are interpreted in terms of General Relativity, where the energy density of the Universe is assumed to be due to non-interacting, dark energy and normal matter. Thus we write

$$\begin{aligned} H^2 = \frac{a_{,\eta}^2}{a^4} &= \frac{\kappa}{3\Phi_0}\rho_{\rm matter} + \frac{\kappa}{3\Phi_0}\rho_{\rm de}^{\rm eff} - \frac{k}{a^2} \\ &= \left(\Omega_{\rm m}^{\rm eff} + \Omega_{\rm de}^{\rm eff}\right)H^2 - \frac{k^2}{a^2}. \end{aligned} \tag{21}$$

The above equation partly defines $\rho_{\rm de}^{\rm eff}$, $\Omega_{\rm m}^{\rm eff}$ and $\Omega_{\rm de}^{\rm eff}$; today $\Phi = \Phi_0$. Now in the Jordan frame: $\rho_{\rm matter} \propto a^{-3}$.

If the effective dark energy equation of state parameter, $w^{\rm de}_{\rm eff}$, were constant it would obey: $\rho^{\rm eff}_{\rm de} \propto a^{-3(1+w^{\rm eff}_{\rm de})}$. More generally however the effective dark energy equation of state is then given by:

$$\rho^{\rm de}_{{\rm eff},\eta} = -3\frac{a_{,\eta}}{a}(1+w^{\rm de}_{\rm eff})\rho^{\rm de}_{\rm eff}. \qquad (22)$$

Taking the η-derivative of Eq. (21) we get:

$$\begin{aligned}\left(2\frac{a_{,\eta\eta}}{a^3} - 4\frac{a^2_{,\eta}}{a^4}\right)\frac{a_{,\eta}}{a} &= \frac{\kappa}{3\Phi_0}\rho_{{\rm matter},\eta} \\ &+ \frac{\kappa}{3\Phi_0}\rho^{\rm eff}_{{\rm de},\eta} + 2\frac{k}{a^2}\frac{a_{,\eta}}{a},\end{aligned}$$

and so using the Eq. (22) and $\rho_{\rm matter} \propto a^{-3}$ we have:

$$\frac{2a_{,\eta\eta}}{a^3} - \frac{4a^2_{,\eta}}{a^4} - \frac{2k}{a^2} = -\frac{\kappa}{\Phi_0}\rho_{\rm matter} - \frac{\kappa}{\Phi_0}(1+w^{\rm eff}_{\rm de})\rho^{\rm eff}_{\rm de}.$$

Finally by adding $3H^2$ to both sides and using Eq. (21) we have:

$$\frac{2a_{,\eta\eta}}{a^3} - \frac{a^2_{,\eta}}{a^4} + \frac{k}{a^2} = -\frac{\kappa}{\Phi_0}w^{\rm eff}_{\rm de}\rho^{\rm eff}_{\rm de}.$$

So by rearranging the Friedmann equations we have found that

$$\begin{aligned} w^{\rm eff}_{\rm de}\kappa\rho^{\rm eff}_{\rm de}/\Phi_0 &= -\frac{2a_{,\eta\eta}}{a^3} + \frac{a^2_{,\eta}}{a^4} - \frac{k}{a^2}, \qquad (23) \\ &= \left[\frac{\Phi_{,\eta\eta}}{\Phi a^2} + \frac{\Phi_{,\eta}a_{,\eta}}{\Phi a^3} - \kappa\Phi V(\phi)\right].\end{aligned}$$

By comparing Eqs. (19) and (21) we see that:

$$\frac{\kappa}{\Phi_0}\rho^{\rm eff}_{\rm de} = \frac{\kappa}{\Phi_0}\rho_{\rm matter}\left(\frac{\Phi_0}{\Phi} - 1\right) + \kappa\Phi V(\Phi) - 3\frac{a_{,\eta}}{a^3}\frac{\Phi_{,\eta}}{\Phi}.$$

Therefore,

$$\begin{aligned}(1+w^{\rm eff}_{\rm de})\kappa\rho^{\rm eff}_{\rm de}/\Phi_0 = &\frac{\Phi_{,\eta\eta}}{\Phi a^2} - \frac{2\Phi_{,\eta}a_{,\eta}}{\Phi a^3} \qquad (24) \\ &+ \frac{\kappa}{\Phi_0}\rho_{\rm matter}\left(\frac{\Phi_0}{\Phi} - 1\right).\end{aligned}$$

Thus, using $3\Omega^{\rm eff}_{\rm de}H^2 = \kappa\rho^{\rm eff}_{\rm de}/\Phi_0$, we have:

$$\begin{aligned}\left(1+w^{\rm eff}_{\rm de}\right)\Omega^{\rm eff}_{\rm de} = &\left[\frac{\Phi_{,\eta\eta}}{3\Phi a^2H^2} - \frac{2\Phi_{,\eta}}{3\Phi aH}\right. \qquad (25) \\ &\left. + \left(\frac{\Phi_0}{\Phi} - 1\right)\Omega^{\rm eff}_{\rm m}\right].\end{aligned}$$

$\left(1+w^{\rm eff}_{\rm de}\right)\Omega^{\rm eff}_{\rm de}$ parametrizes the magnitude of deviations from ΛCDM. If $\phi = -(M_{\rm Pl}/2\beta)\ln\Phi$ has changed by $\Delta\phi$ in the last Hubble time, Eq. (25) implies that, in the recent past and in the Jordan frame, to within an order of magnitude:

$$\left|\left(1+w^{\rm eff}_{\rm de}\right)\Omega^{\rm eff}_{\rm de}\right| \sim \mathcal{O}\left(\frac{\beta|\Delta\phi|}{M_{\rm Pl}}\right). \qquad (26)$$

For later use we rewrite Eq. (25) in terms of $p = \ln a$:

$$\begin{aligned}\left(1+w^{\rm eff}_{\rm de}\right)\Omega^{\rm eff}_{\rm de} = &\frac{\frac{2\Phi_{pp}}{\Phi} + \left(2\Omega^{\rm eff}_{\rm de} - \Omega^{\rm eff}_{\rm m} - 4\right)\frac{\Phi_p}{\Phi}}{3(2+\frac{\Phi_p}{\Phi})} \\ &+ \frac{2\left(\frac{\Phi_0}{\Phi} - 1\right)\Omega^{\rm eff}_{\rm m}}{2+\frac{\Phi_p}{\Phi}}. \qquad (27)\end{aligned}$$

In both the Einstein and Jordan frames, $e^{\frac{\beta(\phi_1-\phi_0)}{M_{\rm Pl}}} - 1$ gives the relative change in the ratio of any particle mass, $m_{\rm p}$, and the Planck mass, $M_{\rm Pl}$ between the times when $\phi = \phi_1$ and when $\phi = \phi_0$. In the Einstein frame $M_{\rm Pl}$ is constant but $m_{\rm p}$ varies whereas in the Jordan frame the converse holds; the ratio of the two masses, being a dimensionless quantity, is the same in either frame. WMAP constrains any such variation in $m_{\rm p}/M_{\rm Pl}$ between now and the epoch of recombination to be $\lesssim 5\%$ at 2σ ($\lesssim 23\%$ at 4σ) [35]. It follows that since recombination

$$|e^{\frac{\beta\Delta\phi}{M_{\rm Pl}}} - 1| < 0.05. \qquad (28)$$

Light element abundances provide similar constraints on any variation in Newton's constant G_N between the present day and the time of nucleosynthesis [36].

Thin-shell constraints, however, provide an even tighter bound on the allowed change in ϕ. To consider these constraints we work in the Einstein frame, however $\Delta\phi$ is the same in either frame.

We assume, as is the case for the real Universe, that the scales of the inhomogeneous regions are small compared to the horizon scale, and that the Universe is approximately homogeneous when coarse-grained over scales larger than some $L_{\rm hom} \ll H^{-1}$. Thus over scales larger than $L_{\rm hom}$, $\phi \approx \phi_b(t)$ and since $L_{\rm hom} \ll H^{-1}$, we can work entirely over sub-horizon scales, which simplifies the analysis greatly. We also assume that the curvature of spacetime is weak over scales smaller than $L_{\rm hom}$. This is equivalent to assuming that the Newtonian potential, U, is small as are the peculiar velocities, v^i, of any matter particles, i.e they are non-relativistic.

Exploiting both the assumption that $HL_{\rm hom} \ll 1$ and that gravity is weak inside the the inhomogeneous regions i.e. $U \ll 1$ and $v^iv^i \ll 1$, we write $\phi = \phi_b(t) + \delta\phi$ and have to leading order in the small quantities and over sub-horizon scales:

$$\nabla^2\delta\phi = V'(\phi) + \frac{\beta\rho}{M_{\rm Pl}} + \ddot{\phi}_b + 3H\dot{\phi}_b.$$

Now

$$-\ddot{\phi}_b - 3H\dot{\phi}_b = V_{,\phi}(\phi_b) + \frac{\beta\rho_b}{M_{\rm Pl}},$$

and so

$$\nabla^2\delta\phi = m_b^2\delta\phi + \frac{\beta\delta\rho}{M_{\rm Pl}} + A(\phi,\phi_b).$$

where

$$A(\phi,\phi_b) \equiv \left[V'(\phi_b+\delta\phi) - V'_b - m_b^2\delta\phi\right].$$

Thus

$$\begin{aligned}\delta\phi = \quad & -\frac{1}{4\pi}\int \mathrm{d}^3x' \frac{e^{-m|x-x'|}}{|x-x'|}\left[\frac{\beta\delta\rho(x',t)}{M_{\rm Pl}}\right. \\ & \left. + A(\phi(x',t),\phi_b(t))\right]. \end{aligned} \quad (29)$$

It is straightforward to show that the condition $V''' < 0$, which must hold for any chameleon theory, implies that $A(\phi,\phi_b) < 0$ for all ϕ and ϕ_b. Thus

$$\delta\phi > -\delta\phi_1 = -\frac{1}{4\pi}\int \mathrm{d}^3x' \frac{e^{-m|x-x'|}}{|x-x'|}\frac{\beta\delta\rho(x',t)}{M_{\rm Pl}}.$$

Now if we require that a test mass at $r=0$ with central density $\rho_c > \rho_b$ has a thin-shell, we must impose that at $r=0$, $\phi \approx \phi_c$, where

$$V_{,\phi}(\phi_c) = -\frac{\beta\rho_c}{M_{\rm Pl}}.$$

Thus ϕ must be able to change by at least $\phi_c - \phi_b = -\Delta\phi_{bc} < 0$ i.e. we have the following necessary condition for thin-shell

$$\frac{\beta\Delta\phi_{bc}}{M_{\rm Pl}} < \frac{\beta\delta\phi_1}{M_{\rm Pl}} = \frac{1}{3}\int \mathrm{d}^3x' \frac{e^{-m|x-x'|}}{|x-x'|} G\delta\rho(x',t). \quad (30)$$

The right hand side of this equation is $\mathcal{O}(U/3)$ or smaller, and the largest values of the peculiar Newtonian potential for realistic models of our Universe are roughly $< 10^{-4}$, and are generally around $10^{-6} - 10^{-5}$ for large clusters and superclusters [26]. Thus we have the following conservative constraint on the cosmological value of the field today:

$$\frac{\beta\Delta\phi_{bc}}{M_{\rm Pl}} < 10^{-4}, \quad (31)$$

We have defined $f(R) = R + h(R)$. The thin-shell constraint certainly ensures that cosmologically today $|\beta\phi/M_{\rm Pl}| \ll 1$ and since we have $1+h'(R) = \exp(-2\beta\phi/M_{\rm Pl})$ by definition we are therefore justified in assuming that we have $|h'(R)| \ll 1$. Then assuming that $|h'| \ll 1$ we find that the potential, $V(\phi)$, is given by:

$$\kappa V(\phi) \approx \frac{1}{2}\left(Rh'(R) - h(R)\right),$$

and

$$-\frac{1}{\beta M_{\rm Pl}}V_{,\phi} \approx R(1-2h'(R)) + 2h(R) - Rh'(R).$$

To leading order then in $|h'(R)|$ we have:

$$-\frac{1}{\beta M_{\rm Pl}}V_{,\phi} \approx R - 4\kappa V(\phi).$$

The chameleon mass squared, $m_\phi^2 = V_{,\phi\phi}$ is then given, to leading order, by:

$$m_\phi^2 = V_{,\phi\phi} \approx \frac{1}{3h''(R)}. \quad (32)$$

Provided $m_\phi^2/H^2 \ll 1$, then the chameleon field will remain close to the minimum of its effective potential [29] cosmological, i.e. $V_{,\phi} = -\beta\rho_{\rm matter}/M_{\rm Pl}$ and the energy density of the chameleon field will be dominated by its potential. Assuming that this is the case we would have:

$$R \approx 4\kappa V(\phi) + \kappa\rho_{\rm matter},$$

and defining $\Omega_m = \kappa\rho_{\rm matter}/3H^2$ and $\Omega_{\rm de} \approx \kappa V(\phi)/3H^2$, we have:

$$R \approx 3(4\Omega_{\rm de} + \Omega_{\rm m})H^2,$$

and so $m_\phi^2/H^2 \gg 1$ becomes:

$$\frac{4\Omega_{\rm de}+\Omega_{\rm m}}{Rh''(R)} \gg 1.$$

Therefore, in many theories, an observationally viable evolution of ϕ requires that it has sat close to the effective minimum of its potential since recombination [29] i.e.:

$$V_{,\phi}(\phi_b(t)) \approx -\frac{\beta\rho_{\rm matter}(t)}{M_{\rm Pl}}$$

Since the background density of matter decreases with time, $V_{,\phi\phi} > 0$ implies that ϕ increases with time. Thus for test mass with density $\rho_c \sim \mathcal{O}(1)\,\mathrm{g\,cm^{-3}}$, we have in the recent past, i.e. out to $z \approx 1$:

$$\phi_c < \phi_b(t) < \phi_b(t_0)$$

where $t = t_0$ is the current time. In this case Eq. (31) gives the following conservative constraint:

$$\frac{\beta}{M_{\rm Pl}}\left(\phi_b(t_0) - \phi_b(t)\right) < 10^{-4},$$

and so, from Eq. (26) we obtain that:

$$|1+w_{\rm eff}|\,\Omega_{\rm de}^{\rm eff} < 10^{-4}. \quad (33)$$

In the recent past where $\Omega_{\rm de}^{\rm eff}$ is not negligible, this leads to a stringent constraint on the deviation of the equation of state from -1. It should be noted that although $|1+w_{\rm eff}|\,\Omega_{\rm de}^{\rm eff}$ is constructed simply out of the scale factor, a, and its derivatives, neither $w_{\rm eff}(z)$ nor $\Omega_{\rm de}^{\rm eff}(z)$ are uniquely defined as functions of redshift in models such as these where the scalar field interacts with normal matter. As a result, it is possible to define $\Omega_{\rm de}^{\rm eff}$ so that it vanishes and even becomes negative in the past. If such a definition is made, then one would (unless the $|1+w_{\rm eff}|\,\Omega_{\rm de}^{\rm eff}$ also happens to vanish) predict that $w_{\rm eff}$ diverges, and hence deviates significantly from -1. A behaviour such as this was noted in Refs. [21, 22]. As a result, an apparent effective deviation from ΛCDM can be deduced.

However, because of the freedom to redefine $\Omega_{\rm de}^{\rm eff}$ and hence $w_{\rm eff}$, one should not rush to assign any physical meaning to the divergence of $w_{\rm eff}$, and deduce that it represents a significant deviation from ΛCDM, since one could always remove this divergence by choosing to define $\Omega_{\rm de}^{\rm eff}$ it such a way that it is positive definite. In all cases, the bound (33) gives an intrinsic measure of the deviation of the background cosmology from ΛCDM, and it all cases it is small. Therefore, the predicted late time cosmology is observationally very close to ΛCDM. Additionally, the prospects for being able to detect such small deviations for ΛCDM at the background level in the near future are poor. Of course, as we have already mentioned, detectably at the perturbative level might be within reach.

This said, the thin-shell constraints do not themselves rule out larger deviations from ΛCDM. It may be that $\beta\phi/M_{\rm Pl}$ has undergone relatively large changes in the past i.e. much larger than $\mathcal{O}(10^{-4})$, but that we now just happen to live at a point in time when $\beta\Delta\phi_{bc}/M_{\rm Pl} < 10^{-4}$. This would, however, be a fairly remarkable coincidence and would inevitably require a great deal of fine-tuning of the theory and the initial conditions. To avoid this new coincidence problem, we would have to require that the cosmological changes in $\beta\phi/M_{\rm Pl}$ have been smaller than $\mathcal{O}(10^{-4})$ in the recent past which would in turn, as we illustrated above, constrain any deviations from ΛCDM to be unobservably small. We note, however, that deviations can be expected on very small scales, as in the original chameleon model [23].

In this section we have sketched how the thin-shell requirement for laboratory test masses place a very strong constraint on the recent cosmological evolution of ϕ, and generally constrains any deviations from ΛCDM in the predicted cosmology to be small. This is not, however, a 'water tight' constraint as it may be possible to circumvent it by requiring a seemingly improbable cosmological evolution wherein such bodies would only have developed thin-shells in the recent (in the cosmological sense) past. The laboratory constraints which we will derive in what follows cannot be avoided in this way.

IV. INVERSE SQUARE LAW CONSTRAINTS

In the weak field limit, the gravitational force due to a small body drops off as $1/r^2$, where r is the distance to the body's centre of mass. If there is an additional scalar degree of freedom to gravity with constant mass m_ϕ, the force instead drops off as:

$$\frac{(1+\alpha(1+m_\phi r)e^{-m_\phi r}))}{r^2},$$

where α parametrizes the strength with which the scalar degree of freedom couples to matter. In $f(R)$ theories $\alpha = 2\beta^2 = 1/3$. When $m_\phi r \ll 1$ or $m_\phi r \gg 1$, the force still drops off, approximately, as $1/r^2$, however there would be a noticeable deviation from this behaviour over scales $r \sim 1/m_\phi$. If, as in chameleon theories, m_ϕ is not a constant but instead undergoes $\mathcal{O}(1)$ or greater variations, the behaviour of the force is more complicated but generally not of inverse square law form.

It is often assumed that what is needed for an $f(R)$ theory to avoid the constraints of inverse square law tests, is that the test bodies develop thin-shells. Generally, however, this is not the case. The presence of a thin-shell causes the chameleonic force due to a body to drop off much faster than $1/r^2$ near the surface of the body. Far from the body, the force has a Yukawa form, although as a result of the fast drop-off near the surface, it is much smaller than one would normally expect it to be. If two thin-shelled bodies are sufficiently close however then they would be inside the region where the faster drop-off is occurring. In these cases the detectable violation of the inverse square law can be much larger than one might expect.

A number of different experiments have searched for violations of the inverse square law. For gravitational strength forces, i.e. $\alpha \sim \mathcal{O}(1)$, the best constraints are currently provided by the Eöt-Wash experiment [32].

The Eöt-Wash experiment [32] consists of two plates: the attractor and the detector. The detector is 0.997 mm thick and made out of molybdenum. The detector has 42 4.767 mm diameter holes bored into it in a pattern with 21-fold azimuthal symmetry. The attractor is similar and consists of a 0.997 mm thick molybdenum plate with 42 3.178 mm diameter, arranged in a pattern with 21-fold azimuthal symmetry, mounted on a thicker tantalum disc with 42 holes, each with diameter 6.352 mm. The holes in the lower tantalum ring are displaced so that the torque on the detector due to the attractor from forces, such as Newtonian gravity, that have a $1/r^2$ behaviour vanishes. The detection of a non-zero torque would therefore indicate the presence of either a correction to gravity with a behaviour different from $1/r^2$ or the presence of a new force that also did not behave as $1/r^2$.

A. Chameleonic Force & Torque

We now calculate the force, due to a chameleonic scalar field, ϕ, on one plate due to the other lying parallel to it. From this we calculate the chameleonic contribution to the torque.

In a background, where $\phi = \phi_b$ far from the plates, the chameleonic force per unit area between two parallel plates, of the same or similar compositions, both with thin-shells and with a distance of separation d between their two facing surfaces was found, under certain conditions, in Ref. [28, 31]. In Appendix A we generalise those formulae. We find that the chameleonic force between two parallel circular plates, with radius r_p and thickness

$D \ll r_p$, and separation $d \ll r_p$ is given by:

$$\frac{F_\phi}{A} = V(\phi_0) - V(\phi_b) - V_b'(\phi_0 - \phi_b) + \frac{(V_b' - V_c')^2}{2m_b^2 C(m_b r_p)} E(\phi_c, \phi_b, m_b r_p). \quad (34)$$

where $\phi_0(d)$ is defined to the values of ϕ midway between the two plates, and formulae for it are provided in Ref. [31]. We have also defined:

$$E = 1 + 2C(m_b r_p) D(\phi_c, \phi_b) - \sqrt{1 + 4CD}, \quad (35)$$

$$C = \frac{(e^{m_b r_p} - 1/2)}{(e^{m_b r_p} - 1)^2}. \quad (36)$$

$$D = \frac{m_b^2 \left[V(\phi_b) - V(\phi_c) - V_{,\phi}(\phi_c)(\phi_b - \phi_c)\right]}{(V_{,\phi}(\phi_b) - V_{,\phi}(\phi_c))^2}. \quad (37)$$

The last term in Eq. (34) represents the only difference between the generalized force formula and the one presented in Refs. ([28]) & ([31]), and we note that the extra term is independent of the separation d. When $C(m_b r_p) D(\phi_c, \phi_b) \ll 1$, the last term in Eq. (34) is negligible. We note that $C(m_b r_p) \ll 1$ when $m_b r_p \gg 1$, and so whenever $m_b r_p \gg 1$, the last term is always negligible.

The details of how $F_\phi(d)/A$ drops off with d will depend on the form of $V(\phi)$. For many choices of $V(\phi)$, e.g. $V(\phi) \propto \phi^n$ for $n < -2$ or $n > 2$, one finds that $F_\phi(d)/A$ drops off faster than $1/d$, for all $d > d_2$ is small compared to both r_p and the radius, r_h, of holes in the plates. Indeed, this will certainly be the case, no matter what form $V(\phi)$ takes, if $m_b r_h \gg 1$ where $m_b = m_\phi(\phi_b)$. Provided this is the case, we can define the potential energy, $\mathcal{V}_\phi(d)$ due to the chameleonic force for two plates with separation $d \ll R_h$ thus:

$$\mathcal{V}_\phi(d) \approx A \int_d^\infty \left(\frac{F_\phi(s)}{A}\right) \mathrm{d}s. \quad (38)$$

The faster $1/d$ drop off has been used to set the upper limit of the above integral to ∞.

In the Eöt-Wash experiment the plates have a number of holes in them. This means that as one plate is rotated, by an angle θ say, relative to another, the surface area, $A(\phi)$, of one plate that faces the other changes. Note that F_ϕ/A does not depend on A. The torque due to the chameleonic force is given by the rate of change of the potential $\mathcal{V}_\phi(d)$ with θ:

$$T_\phi(d) = \frac{\mathrm{d}A}{\mathrm{d}\theta} \int_d^\infty \left(\frac{F_\phi(s)}{A}\right) \mathrm{d}s. \quad (39)$$

We therefore have:

$$T_\phi(d) = a_T \int_d^\infty \left(\frac{F_\phi(s)}{A}\right) \mathrm{d}s, \quad (40)$$

where $a_T = \mathrm{d}A/\mathrm{d}\theta$ is a constant that depends only on the details of the experimental set-up rather than the theory being tested. For the 2006 Eöt-Wash experiment [32] we find

$$a_T = 3.0 \times 10^{-3}\,\mathrm{m}^2.$$

If F_ϕ/A drops off too slowly over scales of the order of r_h then a more complicated analysis must be performed, and knowing the force between two infinite parallel plates is no longer enough to find a good approximation to the torque. Instead a full numerical analysis would have to be undertaken to get accurate results. This said, for $d \gtrsim r_h$, we do not expect $F_\phi(d)$ to depend strongly on θ because the effect of the holes will be largely smeared out over separation distances much larger r_h. On scales $\ll r_h$, we found that $F_\phi \propto A(\phi)$. Since $T_\phi = \mathrm{d}V_\phi/\mathrm{d}\theta$, $\mathrm{d}V_\phi/\mathrm{d}d = F_\phi$, and we expect F_ϕ to be largely θ independent for $d \gg r_h$ and $\propto A(\theta)$ on smaller scales, we expect, to within an order of magnitude, that:

$$T_\phi(d) \approx a_T \int_d^{r_\mathrm{h}} \left(\frac{F_\phi(s)}{A}\right) \mathrm{d}s.$$

in these cases, where once again $a_T = \mathrm{d}A/\mathrm{d}\theta$. By picking r_h as an upper bound for the integral we are probably under estimating the torque as we are dropping the contributions from larger separations.

B. The Effect of an Electrostatic Shield

Up to now we have not considered the role played by the electrostatic shield. Because the shield is so thin ($d_\mathrm{s} = 10\,\mu\mathrm{m}$) compared to the plates but has similar density to the plates, it is safe to say that the shield will only have a thin-shell when the plates have thin-shells. Assuming the plates do have thin-shells, we define m_s to be the mass the chameleon would have deep inside the shield if the shield has a thin-shell i.e. $m_\mathrm{s} = m_\phi(\phi_\mathrm{s})$ where $V'(\phi_\mathrm{s}) = -\beta\rho_\mathrm{s}/M_\mathrm{Pl}$. Since the shield is sandwiched between the two plates, the thin-shell condition for the shield is simply $m_\mathrm{s} d_\mathrm{s} \gtrsim 1$. When the shield has a thin-shell, its presence attenuates the chameleonic force and torque on the detector due to the attractor by a factor of $\exp(-m_\mathrm{s} d_\mathrm{s})$. Since $\exp(-m_\mathrm{s} d_\mathrm{s}) \approx 1$ in the absence of a thin-shelled shield, we can take account of the shield, thin-shell or not, by changing the definition of T_ϕ thus:

$$T_\phi(d) \approx a_T e^{-m_\mathrm{s} d_\mathrm{s}} \int_d^{R_h} \left(\frac{F_\phi(s)}{A}\right) \mathrm{d}s. \quad (41)$$

This expression provides a very good approximation for theories in which the precise value of r_h is unimportant (e.g. ones with $m_b r_h \gg 1$) and an order of magnitude estimate otherwise.

C. Inverse Square Law Constraints

The 2006 Eöt-Wash experiments requires that

$$|T_\phi(d = 55\,\mu\mathrm{m})| < 0.87 \times 10^{-17}\,\mathrm{Nm},$$

with 95% confidence. We define $T_\phi(d=55\mu\,\mathrm{m}) = a_T\Lambda_T^3$ and find that the above bounds correspond to:

$$\Lambda_T < 0.89\times 10^{-12}\,\mathrm{GeV}, \tag{42}$$

Importantly this is smaller than the energy scale associated with dark energy: $\Lambda_{\mathrm{de}} = 2.4\times 10^{-12}\,\mathrm{GeV}$; $\rho_{\mathrm{de}} = \Lambda_{\mathrm{de}}^4$.

Using our expression Eq.(41) for the chameleonic torque, we find that the constraints we must apply are as follows:

$$e^{-m_{\mathrm{s}} d_{\mathrm{s}}}\int_{55\,\mu\mathrm{m}}^{R_h}\left(\frac{F_\phi(s)}{A}\right)\mathrm{d}s \;<\; 7.0\times 10^{-37}\,\mathrm{GeV}^3. \tag{43}$$

V. APPLICATION OF CONSTRAINTS TO $f(R)$ THEORIES

A. Chameleonic force

The chameleonic force per unit area between two parallel plates is given by Eq. (34). To prevent large deviations from general relativity occurring over solar system, and smaller, scales, one must require that $f(R)\approx R+h(R)$ where $|h'(R)|\ll 1$ and $|h(R)/R|\ll 1$. In this case the expression for F_ϕ/A becomes:

$$\frac{F_\phi}{A}\approx\frac{M_{\mathrm{Pl}}^2}{2}\left[(R_0-R_b)\,h'(R_0)+(h(R_b)-h(R_0))+\mathcal{F}_0\right].$$

where for $R_c\gg R_b$

$$\mathcal{F}_0 = \frac{R_c^2 h''(R_b)}{4C(m_b r_{\mathrm{p}})}\mathcal{E}_0, \tag{44}$$

$$\mathcal{E}_0 = 1+2C(m_b r_{\mathrm{p}})\mathcal{D}_0(R_c,R_b)-\sqrt{1+4C\mathcal{D}_0}, \tag{45}$$

$$\mathcal{D}_0 = \frac{h(R_c)-h(R_b)-(R_c-R_b)h'(R_b)}{h''(R_b)R_c^2}. \tag{46}$$

where r_{p} is the radius of the parallel plate(s). We shall now consider several potential forms for $h(R)$.

B. Logarithmic potentials

We begin by considering a simple chameleon gravity model that was recently suggested in Ref. [37] for a general β. The theory, when written as a chameleon theory, would have a potential $V(\phi) = V_0 - \Lambda_0^4\ln(\phi/M_{\mathrm{Pl}})$, it was suggested that this would result in an experimentally viable and cosmologically interesting dark energy model, where $\Lambda_0^4/M_{\mathrm{Pl}}^2\sim\mathcal{O}(H_0^2)$ [37] for $\beta\le 1/4\sqrt{3}$. We will analyse the same model in the $f(R)$ setting where $\beta = 1/\sqrt{6} > 1/4\sqrt{3}$ and show that local tests already lead to difficulties, see also [38].

On laboratory scales we would have $f(R)\approx 1+h(R)$ and so we find:

$$h(R) = -\frac{2\Lambda_0^4}{M_{\mathrm{Pl}}^2}\left[\frac{V_0}{\Lambda_0^4}+\ln(2\beta)+\ln\left(\frac{M_{\mathrm{Pl}}^2 R}{2\Lambda_0^4}\right)+1\right].$$

Assuming that $m_b\ll m_c$ where m_c is the chameleon mass inside the plates, it follows that F_ϕ/A has the following form:

$$\frac{F_\phi}{A} = \Lambda_0^4\left[\ln\left(\frac{R_0}{R_b}\right)+\frac{R_b}{R_0}-1\right]+\frac{1}{2}\Lambda_0^4\frac{m_c^2\mathcal{E}_0}{m_b^2 C(m_b r_{\mathrm{p}})}. \tag{47}$$

where

$$\mathcal{E}_0 = 1+\frac{2C(m_b r_{\mathrm{p}})m_b}{m_c}-\sqrt{1+\frac{4C(m_b r_{\mathrm{p}})m_b}{m_c}}.$$

When $C(m_b r_{\mathrm{p}})m_b/m_c\ll 1$ we have $\mathcal{E}_0\approx 2C^2m_b^2/m_c^2$ and so the last term in Eq. (47) is:

$$\Lambda_0^4\frac{m_c^2}{2m_b^2C(m_b r_{\mathrm{p}})}\mathcal{E}_0\approx\Lambda_0^4 C(m_b r_{\mathrm{p}}).$$

Alternatively if $C(m_b R)m_b/m_c\gg 1$ we would have:

$$\Lambda_0^4\frac{m_c^2}{2m_b^2C(m_b r_{\mathrm{p}})}\mathcal{E}_0\approx\Lambda_0^4\frac{m_c}{m_p}.$$

If $C(m_b r_{\mathrm{p}})\ll 1$ i.e. $m_b r_{\mathrm{p}}\gg 1$, then it is clear that this last term is always small compared to the other terms, however if $m_b r_{\mathrm{p}}\ll 1$, then the last term will dominate the expression for the force.

The chameleon mass for a given R in this set-up is

$$m_\phi(R) = \frac{M_{\mathrm{Pl}}R}{\sqrt{6}\Lambda_0^2}.$$

In between the two plates, ϕ satisfies [31]:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = V_{,\phi}(\phi)-V_{,\phi}(\phi_b),$$

and ϕ_0 is defined to be value of ϕ midway between the two plates (i.e. a distance $d/2$ from either plate), where by symmetry $\mathrm{d}\phi/\mathrm{d}z = 0$. Integrating the above equation we therefore have:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 = 2(V(\phi)-V(\phi_0)-V_{,\phi}(\phi_b)(\phi-\phi_0)).$$

Integrating this again and defining $\phi_{\mathrm{s}}\sim\mathcal{O}(\phi_c)$ to be the value of ϕ on the surface of the plates, we have:

$$\frac{d}{\sqrt{2}} = \int_{\phi_{\mathrm{s}}}^{\phi_0}\frac{\mathrm{d}x}{\sqrt{V(x)-V(\phi_0)-V_{,\phi}(\phi_b)(\phi-\phi_0)}}.$$

Following Ref. [31], when $m_b\ll m_0\ll m_c$ we have that $\phi_s\sim\mathcal{O}(\phi_c)\ll\phi_0\ll\phi_b$ and so $V(\phi)-V(\phi_0)-V_{,\phi}(\phi_b)(\phi-\phi_0)\approx V(\phi)-V(\phi_0) = \Lambda_0^4\ln(\phi_0/\phi)$ and so:

$$\frac{d}{\sqrt{2}}\approx\frac{\phi_0}{\Lambda_0^2}\int_0^1\frac{\mathrm{d}x}{\sqrt{\ln(1/x)}}.$$

Noting that $m_0 = \Lambda_0^2/\phi_0$ and evaluating the integral we find:

$$m_0 d = \frac{M_{\mathrm{Pl}}R_0 d}{\sqrt{6}\Lambda_0^2} = \sqrt{2\pi}.$$

Now $m_b \ll m_0 \ll m_c$ is clearly equivalent to $R_c \gg R_0 \gg R_b$, and in these cases we therefore have:

$$R_0 = \frac{2\sqrt{3\pi}\Lambda_0^2}{M_{\rm Pl} d}. \tag{48}$$

It follows that, irrespective of the value of $\mathcal{E}_0$, F_ϕ/A drops off more slowly than $1/d$ for all $m_b d \ll 1$. In the latest version of the Eöt-Wash experiment [32], the plate radius, $r_{\rm p}$, is 3.5 cm, and the smallest hole radius is 1.6 mm. The pressure of the laboratory vacuum is 10^{-6} torr which corresponds to a background density of $\rho_b = 1.6\times 10^{-9}\,{\rm kg\,m^{-3}} = 6.7\times 10^{-30}\,{\rm GeV}^4$.

Now if the vacuum region is large enough then $R_b = \rho_b/M_{\rm Pl}^2$ and so $m_b = \bar{m}_b \equiv \rho_b/\sqrt{6}\Lambda_0^2 M_{\rm Pl}$. However, it was shown in Refs. [28, 30, 31] that if the vacuum region only have length scale $L_{\rm vac}$ and $\bar{m}_b \ll 1/L_{\rm vac}$, then generically $m_b \sim \mathcal{O}(1/L_{\rm vac})$. For the moment we only assume that $r_{\rm p}/L_{\rm vac} \ll 1$.

We therefore find that for $d = 55\,\mu$m we have $m_b d < 1$ for all $\Lambda_0 > 5.6\times 10^{-19}$ GeV, $m_b r_{\rm h} < 1$ for $\Lambda_0 > 3.0\times 10^{-18}$ GeV and $m_b r_{\rm p} < 1$ for $\Lambda_0 > 1.4\times 10^{-17}$ GeV. The suppression factor due to the electrostatic shield is $\exp(-m_s d_{\rm s})$ where:

$$m_{\rm s} d_{\rm s} = \frac{\beta \rho_{\rm shield} d_{\rm s}}{M_{\rm Pl}\Lambda_0^2} = 0.30\left(\frac{10^{-12}\,{\rm GeV}}{\Lambda_0}\right)^2,$$

where we have used $d_{\rm s} = 10\,\mu$m and $\rho_{\rm s} = 8.3\,{\rm g\,cm^{-3}}$. Thus, whenever $m_{\rm s} d_{\rm s} \lesssim 1$, we are therefore firmly in the $m_b r_{\rm p}\, m_b r_{\rm h} \ll 1$ region and hence $C(m_b r_{\rm p}) \gg 1$. Thus $C(m_b r_{\rm p}) \approx 1/2 m_b^2 r_{\rm p}^2$ and:

$$\begin{aligned}\frac{F_\phi}{A} \approx\ & \Lambda_0^4\left[-\ln(m_b d/\sqrt{2\pi}) + m_b d/\sqrt{2\pi} - 1\right] \\ & + \Lambda_0^4 m_c^2 r_{\rm p}^2 \mathcal{E}_0\end{aligned}$$

From Eq.(41), in the absence of the electrostatic shield, the chameleonic torque for $r_h \gg d$ would be:

$$\begin{aligned}T_\phi \approx\ & a_T \Lambda_0^4 r_h \left[\ln\left(\frac{\sqrt{2\pi}}{m_b r_h}\right) + \frac{m_b r_h}{\sqrt{2\pi}}\right. \\ & \left. + m_c^2 r_{\rm p}^2 \mathcal{E}_0\right].\end{aligned} \tag{49}$$

Here m_c is the chameleon mass inside the plates which have density $\rho_c \approx 10.2\,{\rm g\,cm^{-3}}$.

We note that the requirement that the plates have a thin-shell constrains the value of m_b, and it is important to check that this constraint holds. Conservatively, the thin-shell constraints for the plate require $\beta(\phi_b - \phi_c)/M_{\rm Pl} < \Phi_N/3$ where Φ_N is the Newtonian gravitational potential of the whole experiment at the surface of the plate. Since the geometry of the experiment is complicated, we do not calculate Φ_N. Instead, we estimate $\Phi_N/3 \lesssim 10^{-26}$, and so $\phi_b - \phi_c \lesssim 7\times 10^{-8}$ GeV. Given that $\rho_c \gg \rho_b$, we take $\phi_c \ll \phi_b$ and then from $m_b = \Lambda_0^2/\phi_b$ we must have

$$1/m_b < 14\,{\rm m}\left(\frac{10^{-12}\,{\rm GeV}}{\Lambda_0}\right)^2. \tag{50}$$

If this condition does not hold, then the plates would not have thin-shells and the Eöt-Wash data would automatically rule out the theory. The experiment takes place inside a vacuum chamber with smallest dimension $L_{\rm vac} = 0.2$ m [33]. We assume that the walls of the vacuum chamber have thin-shells. Approximating the walls of the vacuum chamber perpendicular to the shortest dimension as being parallel plates, we use Eq. (48) above to tell us that when the background density of matter in the vacuum chamber is very small, we have in the centre of the chamber:

$$R_b = \bar{R}_{\rm vac} = \frac{2\sqrt{3\pi}\Lambda_0^2}{M_{\rm Pl} L_{\rm vac}}.$$

This formula holds as long as $\bar{R}_b = \rho_b/M_{\rm Pl}^2 \lesssim \bar{R}_{\rm vac}$. In the opposite limit we just have $R_b = \bar{R}_b$. In all cases we have $R_b \geq \bar{R}_{\rm vac}$ and so:

$$m_b \geq \frac{M_{\rm Pl}\bar{R}_{\rm vac}}{\sqrt{6}\Lambda_0^2} = \frac{\sqrt{2\pi}}{L_{\rm vac}} \approx 12\,m^{-1} \tag{51}$$

and so condition (50) is always satisfied for $\Lambda_0 \lesssim 1.2\times 10^{-11}$ GeV.

Given Eq. (51), we find that for the allowed values of Λ_0 we always have:

$$\frac{C(m_b r_{\rm p}) m_b}{m_c} \approx \frac{1}{2 m_b m_c r_{\rm p}^2} \ll 1,$$

where we have used $m_c = \rho_c/\sqrt{6} M_{\rm Pl}\Lambda_0^2$. Thus:

$$\mathcal{E}_0 \approx 2C^2 m_b^2/m_c^2 = (m_c m_b r_{\rm p})^{-4}/2.$$

It follows that, in the absence of the electrostatic shield, the chameleonic torque for $r_h \gg d$ is

$$\begin{aligned}T_\phi \approx\ & a_T \Lambda_0^4 r_h \left[\ln\left(\sqrt{2\pi} m_b r_h\right) + \frac{m_b r_h}{\sqrt{2\pi}}\right. \\ & \left. + \frac{1}{2 m_b^2 r_{\rm p}^2}\right].\end{aligned} \tag{52}$$

Including the suppression factor due to the electrostatic shield, which is $\exp(-m_s d_s)$, we therefore find the following constraint on $y_0 = \Lambda_0/(10^{-12}\,{\rm GeV})$:

$$e^{-0.30 y_0^{-2}/4} y_0 < 0.21,$$

which gives $y_0 < 0.37$ and so:

$$\Lambda_0 < 3.7\times 10^{-13}\,{\rm GeV}. \tag{53}$$

Cosmologically, the mass of the scalar field at the minimum of its potential is given by

$$m_{\rm cos} = \left[\sqrt{\frac{3}{2}}\frac{\Omega_m M_{pl} H}{\Lambda_0^2}\right] H,$$

and the value of ϕ at this minimum is given by:

$$\frac{\beta\phi_{\rm cos}^{\rm min}}{M_{\rm Pl}} = \frac{\Lambda_0^4}{3\Omega_m H^2 M_{\rm Pl}^2}.$$

Now $H = 2.1h \times 10^{-42}\,\mathrm{GeV}$ and from WMAP [39]: $\Omega_m = 0.127h^{-2}$ and $h = 0.73$. We find that

$$m_{\rm cos}/H = \frac{1.1}{y_0^2}, \qquad (54)$$

$$\frac{\beta\phi_{\rm cos}^{\rm min}}{M_{\rm Pl}} = 0.06y_0^4 \qquad (55)$$

and so the Eöt-Wash constraint on Λ_0 gives:

$$m_{\rm cos}/H > 8.$$

At its minimum then, the ϕ-field is still heavy today. This should be contrasted with the requirement obtained in [37] that the mass of the ϕ-field should be small compared to the Hubble rate in order to drive acceleration. Here we find that local tests and the thin shell requirement impose that the mass of the ϕ-field at the cosmological minimum is so large that the field must sit there on cosmological scales. It is easily checked that $m_{\rm cos}/H$ is a decreasing function of time, and so in the past ϕ was heavier still relative to H. Therefore ϕ will have remained stuck close to the minimum of its effective evolution throughout the matter era. Additionally the Eöt-Wash constraint on Λ_0 implies that:

$$\frac{\beta\phi_{\rm cos}^{\rm min}}{M_{\rm Pl}} < 0.001$$

We have considered the potential $V(\phi) = V_0 - \Lambda_0^4 \ln(\phi/M_{\rm Pl})$. The constraint on Λ_0 implies that $\Lambda_0^4/3M_{\rm Pl}^2H^2 < 0.00026$ and so if $V(\phi)$ is to be the source of dark energy and there is to be a realistic amount of it today, we would need $V_0 \gtrsim \mathcal{O}(1000)\Lambda_0^4$. Note that this is very different from the original scenario envisaged in ref. [37], where $V_0 \sim \Lambda_0^4$ so that the whole potential could be written in the form $V = -\Lambda_0^4 \ln(\phi/M)$ where $M \sim \mathcal{O}(M_{\rm Pl})$. The Eöt-Wash constraint on Λ_0 therefore rules out a scenario where $V_0 \sim \Lambda_0^4$ for $\beta = 1/\sqrt{6}$, confirming the cosmological obstruction noted when $\beta > 1/4\sqrt{3}$. If we moved away from $f(R)$ theories and allowed for different couplings, we would find similar constraints on Λ_0 for other $\mathcal{O}(1)$ values of the coupling β.

Relaxing the constraint $V_0 \sim \Lambda_0^4$ and allowing much smaller values of Λ_0, it should also be noted that the conservative thin-shell constraint for a test mass with density $\gg \Lambda_0^4$ on the cosmological value of ϕ (as derived in Section III) actually provides a stronger constraint on the cosmological value of the field today and as such gives a tighter bound on Λ_0. Specifically Eq. (31) implies:

$$\frac{\beta\phi_{\rm cos}}{M_{\rm Pl}} < 10^{-4} \Leftrightarrow \Lambda_0 < 1.8 \times 10^{-13}\,\mathrm{GeV}. \qquad (56)$$

This leads to the following constraint on the mass of ϕ at its minimum cosmologically:

$$m_{\rm cos}/H > 35. \qquad (57)$$

Since $m_{\rm cos}/H \gg 1$, ϕ lies close to its cosmological minimum and so, in the Jordan frame, by Eq. (18):

$$-2\Phi^3 V_{,\Phi} = \frac{M_{\rm Pl}}{\beta}\Phi^2 V_{,\phi} = -\rho_{\rm matter},$$

where $\Phi = e^{-2\beta\phi/M_{\rm Pl}}$ and so $\Phi \approx 1$. To leading order with $p = \ln a$ we have $\phi_p \approx 3\phi$. Therefore

$$\frac{\Phi_p}{\Phi} = -2\frac{\beta\phi_p}{M_{\rm Pl}} \approx \frac{-6\beta\phi}{M_{\rm Pl}} = -\frac{6\Lambda_0^4}{\rho_{\rm matter}},$$

and to the same order

$$\frac{\Phi_{pp}}{\Phi} \approx -\frac{18\beta\phi}{M_{\rm Pl}} = -\frac{18\Lambda_0^4}{\rho_{\rm matter}}.$$

We define $\theta = \Lambda_0^4/\rho_{\rm matter}$ and then using Eq. (27) we arrive at

$$(1 + w_{\rm eff})\Omega_{\rm de}^{\rm eff} \approx \frac{(2\Omega_{\rm de}^{\rm eff} - \Omega_{\rm m}^{\rm eff} + 2)\theta}{3\theta - 1} + \frac{(\theta - \theta_0)\Omega_{\rm m}^{\rm eff}}{1 - 3\theta} \qquad (58)$$

where θ_0 is the value of θ at the present time. Assuming that the Universe is flat ($k = 0$) and taking $\Omega_{\rm de}^{\rm eff} = 0.76$, we find today when $t = t_0$:

$$1 + w_{\rm eff}(t_0) \approx -4.32\theta_0.$$

Notice that the effective equation of state is below -1, this is a consequence of the scalar-tensor character of the chameleon model.

The Eöt-Wash constraint on Λ_0 gives

$$|1 + w_{\rm eff}| < 0.0085,$$

while the thin-shell constraint on Λ_0 gives

$$|1 + w_{\rm eff}| < 10^{-4},$$

which is in line with our expectations from Section III. Whilst the thin-shell constraint on the cosmology is much stronger than the Eöt-Wash bound, the cosmological constraint makes a number assumptions above the nature of inhomogeneities in the Universe, in particular about their scale at the present time. One could presumably argue that the cosmological constraint could be relaxed. The same line of argument cannot be used for the Eöt-Wash constraint, and as such represents a strong constraint on the magnitude of deviations from ΛCDM in this model. As such, the model cannot be distinguished from a ΛCDM model at the background level. At the perturbative level, the situation is very different as the bound (57) implies that density contrast would have an anomalous growth on scales lower than $100h^{-1}$Mpc. This may be testable in the near future with next generation red-shift surveys[25].

The version of the logarithmic potential $f(R)$ theory suggested in Ref. [37] required $\Lambda_0^4 \geq \rho_{\rm matter}$ today i.e.

$\Lambda_0 > 1.7 \times 10^{-12}$ GeV. Even from a conservative point of view (and indeed for any β where $2\beta^2 \sim \mathcal{O}(1)$), such a value of Λ_0 would produce a torque in the Eöt-Wash experiment that is almost 100 times larger than the 95%-confidence level upper bound. The scenario suggested in Ref. [37] is therefore strongly ruled out by local tests of gravity.

C. Power-law form

In many cases [24, 26, 34] one finds that for $R \gg H_0^2$, where H_0^2 is the Hubble constant today, $h(R)$ has a power law form i.e.:

$$h(R) \approx \frac{p}{p+1}\bar{R}\left(\frac{R}{\bar{R}}\right)^{p+1}, \tag{59}$$

for some $p \neq 0$ and some constant $\bar{R} > 0$. For a chameleon mechanism to exist we need $V' < 0$, $V'' > 0$ and $V''' < 0$, and so must require $p < 1$. Relative divergences from GR such as those parametrized in the PPN formalism or measured by observing the motions of planets would scale as $h(R)/R$ and $h'(R)$ or by the ratio of any variable component of the effective cosmological constant to the local matter density . However, the Eöt-Wash test probes changes in $V(\phi)$, which scales as $h(R)$ and $Rh'(R)$, although they are only sensitive to this when the chameleon mass in the background, which scales as $1/h''(R)$, is not too large.

In theories with $0 < p < 1$, both $h(R)/R$ and $V(\phi)$ would be largest for large values of R. These theories would therefore diverge most markedly from General Relativity in the UV (i.e. large R) regime. Increasing $\bar{R}$ would make both $h(R)/R$ and $h''(R)$ smaller, and so ultimately one could ensure compatibility with all laboratory tests by making $\bar{R}$ very large. Provided h'' is not small, however, the changes in $V(\phi)$ that could be detected by the Eöt-Wash experiment would increase.

If $-1 < p < 0$ then $h(R)/R$ and $h'(R)$ are largest in the IR regime where R is small. However $V(\phi)$ still increases with R, and since R increases as the separation of the plates in the Eöt-Wash experiment is decreased, the smaller the separations the stronger the potentially detectable signal would be. Ultimately compatibility with all local tests could be ensured by making $\bar{R}$ small enough. Additionally in all theories where $p > -1$, F_ϕ/A would be dominated by the d-dependent (i.e. R_0 dependent) terms and only weakly depend on R_b when $m_b d \ll 1$.

Finally, in theories with $p < -1$ both $V(\phi)$ and $h(R)/R$ would decrease with R. This would mean that F_ϕ/A would only very weakly depend on d and generally be much smaller in a given set-up than for the other classes of theories. Again compatibility with all local tests could be ensured by making $\bar{R}$ small enough.

The $-1 < p < 0$ theories are the most testable type of theory as they would result in deviations from GR in both the UV and IR regimes. In the UV regime there would be potentially detectable fifth-forces between parallel plates, and in the IR regime the ratio of the density dependent part of the effective cosmological constant to the ambient matter density would increase cosmologically at late times as the ambient density decreased.

In all of these theories:

$$\begin{aligned}\frac{F_\phi}{A} = {} & \frac{M_{\rm Pl}^2 R_0 p^2}{2(p+1)}\left(\frac{R_0}{\bar{R}}\right)^p \\ & + \frac{M_{\rm Pl}^2 R_b p}{2(p+1)}\left(\frac{R_b}{\bar{R}}\right)^p - \frac{M_{\rm Pl}^2 R_b p}{2}\left(\frac{R_0}{\bar{R}}\right)^p \\ & + \frac{M_{\rm Pl}^2 p^2 R_c^2}{8C(m_b r_{\rm p})R_b}\left(\frac{R_b}{\bar{R}}\right)^p \mathcal{E}_0.\end{aligned} \tag{60}$$

where $\mathcal{E}_0$ is given in terms of C and $\mathcal{D}_0$ by Eq. (45) and

$$\mathcal{D}_0 = \frac{1}{p(p+1)}\left[\left(\frac{R_b}{R_c}\right)^{1-p} + p\left(\frac{R_b}{R_c}\right)^2 - (p+1)\left(\frac{R_b}{R_c}\right)\right].$$

Note that the last term in Eq. (60) is independent of the plate separation d and vanishes in the limit $m_b r_{\rm p} \to \infty$.

1. Relationship to chameleon theories

Converting these theories to chameleon theories we have for $h' \ll 1$:

$$-2\frac{\beta\phi}{M_{\rm Pl}} = h'(R) = p\left(\frac{R}{\bar{R}}\right)^p,$$

and so $R \propto \phi^{1/p}$ and $h(R)$, $Rh'(R) \propto \phi^{(p+1)/p}$. It follows that

$$V(\phi) = \text{const} + \frac{p^2 M_{\rm Pl}^2 \bar{R}}{2(p+1)}\left(\frac{-2\beta\phi}{M_{\rm Pl} p}\right)^{\frac{p+1}{p}}.$$

and so defining $n = -(p+1)/p$, we see that, neglecting the constant term in the potential, $|V(\phi)| \propto |\phi|^{-n}$. In the context of chameleon theories these potentials have been studied in great detail [16, 28, 29, 30, 31], and so we are able to apply a raft of results to the analysis of these theories.

In appendix B, we show the mass of the chameleon field for $m_c \gg m_0 \gg m_b$ (m_c is the chameleon mass deep inside the plates and m_b is the chameleon mass in the background) is given by $m_0 d = a_p$ where:

$$\begin{aligned} a_p &= \sqrt{\frac{2}{1+p}}p^2 B\left(\frac{1}{2}, \frac{p}{(1+p)}\right) \qquad p \le -1, \\ a_p &= \sqrt{\frac{2}{1+p}} B\left(\frac{1}{2}, \frac{(1-p)}{2(1+p)}\right) \quad -1 \le p \le 1. \end{aligned}$$

Using $m_0 \approx 1/3h''(R_0)$ we therefore have when $R_b \ll R_0 \ll R_c$:

$$\frac{R_0}{\bar{R}} = \left(\frac{3p^2 a_p^2}{\bar{R} d^2}\right)^{\frac{1}{1-p}}. \tag{61}$$

For what follows we also define:

$$K_p = \left(3p^2 a_p^2\right)^{\frac{1+p}{1-p}}. \tag{62}$$

Using $m_0 \approx 1/3h''(R_0)$ we therefore have when $R_b \ll R_0 \ll R_c$:

Using the relationship between R_0 and d derived above when $R_c \gg R_0 \gg R_b$ Eq. (60) becomes:

$$\begin{aligned}\frac{F_\phi}{A} &\approx \frac{M_{\rm Pl}^2 K_p p^2 \bar{R}}{2(p+1)}\left(\frac{1}{\bar{R}d^2}\right)^{\frac{1+p}{1-p}} G_p\left(\frac{m_b d}{a_p}\right), \\ &+ \frac{M_{\rm Pl}^2 p^2 R_c^2}{8C(m_b r_{\rm p})R_b}\left(\frac{R_b}{\bar{R}}\right)^p \mathcal{E}_0.\end{aligned} \tag{63}$$

where

$$\begin{aligned}G_p\left(\frac{m_b d}{a_p}\right) = &\quad 1+\frac{1}{p}\left(\frac{m_b d}{a_p}\right)^{\frac{2(1+p)}{1-p}} \\ &-\frac{p+1}{p}\left(\frac{m_b d}{a_p}\right)^{\frac{2}{1-p}}.\end{aligned} \tag{64}$$

We note that $G_p = 0$ when $m_b d/a_p = 1$ which corresponds to $R_0 = R_b$. We now consider the integral:

$$I(d, r_{\rm h}) = \int_d^{r_{\rm h}} \frac{F_\phi(s)}{A} {\rm d}s$$

The approximation used to calculate $R_0(d)$ breaks down when $R_0 \approx R_b$, which corresponds to $m_b d/a_p \approx 1$. In the case $m_b r_{\rm h}/a_p > 1$ we cannot simply use Eq. (63) to calculate $I(d, r_{\rm h})$ as we must integrate over values of d for which Eq. (63) is not valid. However we should be able to trust Eq. (63) for smaller values of d. For $m_b d/a_p \gg 1$ we expect an exponential drop-off in the force, just as one would find in a Yukawa theory at distances larger than the inverse mass of the scalar field. We therefore do not expect the dominant contribution to $I(d, r_{\rm h})$ to come from values of $d < a_p/m_b$. We also note that if $m_b r_{\rm h}/a_p \gg 1$ then $m_b r_{\rm p} \gg 1$ as $r_{\rm h} < r_{\rm p}$ and as such the second term in Eq. (63) is negligible. The first term in Eq. (63) vanishes when $m_b d/a_p = 1$, and since we do not expect a significant contribution to the integral to come from larger separations, we evaluate $I(d, r_{\rm h})$ by using Eq. (63) for $F_\phi(s)/A$ but if $m_b r_{\rm h}/a_p$ we cut the integral off at a separation $a_p/m_{\rm h}$.

Thus we define $x(d) = m_b d/a_p$ and $x_{\rm max} = \min(m_b r_{\rm h}/a_p, 1)$ and find:

$$\begin{aligned}I(d, r_{\rm h}) \approx &\ \frac{p^2 M_{\rm Pl}^2 K_p \bar{R}^{1/2}}{2}\left(\frac{m_b}{\bar{R}^{1/2} a_p}\right)^{\frac{1+3p}{1-p}} \\ &\times \{[H_{\rm p}(x(d)) - H_{\rm p}(x_{\rm max})] \\ &+ \frac{R_c^2(x_{\rm max}-x)}{4R_b^2 C(m_b r_{\rm p})}\mathcal{E}_0\right\},\end{aligned} \tag{65}$$

where

$$\begin{aligned}H_{\rm p}(x) = &\ \frac{1}{(1+p)}\left[\frac{1-p}{1+3p}x^{-\frac{1+3p}{1-p}} - \frac{x}{p}\right. \\ &\left.+\frac{1-p^2}{p(1-3p)}x^{\frac{1-3p}{1-p}}\right].\end{aligned}$$

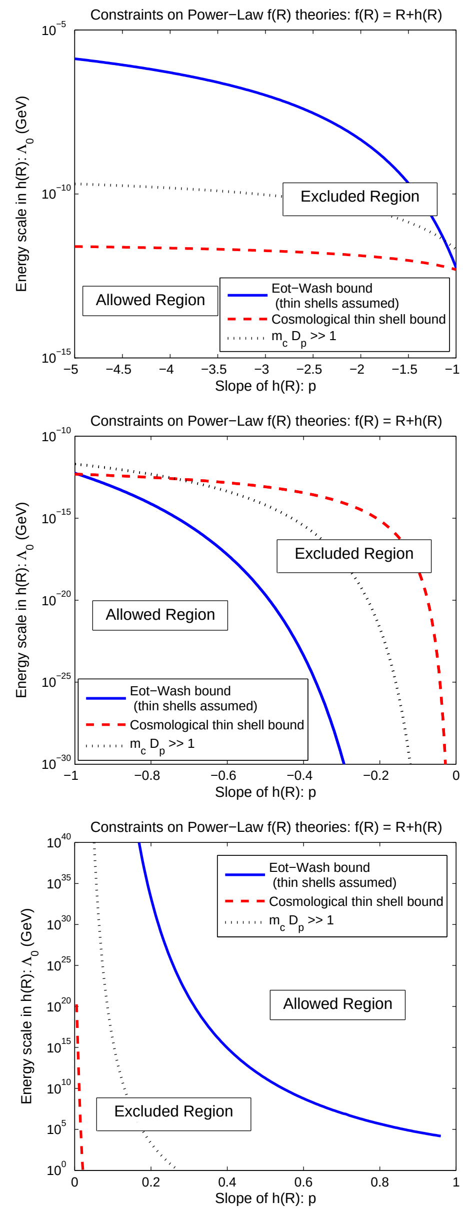

FIG. 1: Eöt-Wash constraints (thick solid blue line) on $f(R)$ gravity theories with $f(R) = R + h(R)$ where $h(R) = \bar{R}(R/\bar{R})^{p+1}$; $\bar{R} = \Lambda_0^4/M_{\rm Pl}^2$ and $-5 < p < -1$, $-1 < p < 0$ and $0 < p < 1$. For this constraint we have assumed that the test bodies have thin-shells (which is necessary to avoid local tests). We have also shown : (1) the cosmological thin shell constraint (thick red dashed line) for test bodies in the laboratory derived in Section III, (2) the naive constraint (thick black dotted line) one could derive by simply requiring that, inside the test bodies, the mass of the chameleon at the minimum of its effective potential, $m_c$is large compared with the length scale of the body, D_p. This was the constraint considered in Ref. [34]. For all such theories we see that the correctly evaluated constraint provided by the Eöt-Wash experiment [32] is stronger than both this naïve constraint and the cosmological thin-shell bound for all for $p \gtrsim -1$. The $m_c D_{\rm p} \gg 1$ constraint never provides the strongest constraint.

With this formula we are able to evaluate the Eöt-Wash constraint for all theories with $h(R) \propto R^{p+1}$. We do this further below. However, we discuss first the cosmological thin-shell constraint on these theories.

2. Cosmological Constraints

On cosmological scales, the field ϕ is stuck at the minimum of the effective potential provided $m_\phi^2/H^2 \gg 1$ which becomes:

$$\frac{4\Omega_{\rm de} + \Omega_{\rm m}}{Rh''(R)} \gg 1.$$

If $h(R) \propto R^{p+1}$ this becomes:

$$\frac{4\Omega_{\rm de} + \Omega_{\rm m}}{|ph'(R)|} \gg 1. \tag{66}$$

The cosmological thin-shell constraint requires that:

$$\frac{\beta|\Delta\phi|}{M_{\rm Pl}} \lesssim 10^{-4},$$

where $\Delta\phi$ is the difference between the value of ϕ cosmological and the value of ϕ at the minimum of the effective potential in a region with density $\mathcal{O}(1)\,{\rm g\,cm^{-3}}$. This generally implies that cosmologically $\beta|\phi|/M_{\rm Pl} \lesssim 10^{-4}$ and $\frac{1}{2}|h'(R)| \lesssim 10^{-4}$. It is clear then that Eq. (66) holds provided $p \times 10^{-4} \ll 1$ and so for $\mathcal{O}(1)$ values of p, we are always in the region where $m_\phi^2/H^2 \gg 1$ and ϕ lies close to the minimum of its effective potential.

To leading order we take $V_{,\phi} \approx -\beta\rho_{\rm matter}/M_{\rm Pl}$ and, defining $p = \ln a$, where a is the FRW scale factor in the Jordan frame and $\Phi = e^{-2\beta\phi/M_{\rm Pl}}$, we find that:

$$\frac{\Phi_p}{\Phi} \approx -\frac{3\Omega_{\rm m}H^2}{m_\phi^2} = -3f_0Rh''(R) \ll 1, \tag{67}$$

where $f_0 = \Omega_{\rm m}/(\Omega_{\rm m} + 4\Omega_{\rm de})$

$$\frac{\Phi_{pp}}{\Phi} \approx 9f_0Rh''\left[1 + \frac{f_0Rh'''(R)}{h''(R)}\right]. \tag{68}$$

Today from Eq. (27), with $\Omega_m^{\rm eff} = \Omega_m$, $\Omega_{\rm de}^{\rm eff} = \Omega_{\rm de} \approx 1 - \Omega_m$ we have $f_0 \approx \Omega_{\rm m}/(4 - 3\Omega_{\rm m})$:

$$(1 + w_{\rm de}^{\rm eff})\Omega_{\rm de} \approx 3f_0Rh''(R)\left[\frac{4}{3} + \frac{f_0Rh'''(R)}{h''(R)} + \frac{\Omega_{\rm m}}{2}\right], \tag{69}$$

and so for $\Omega_{\rm m} = 0.24$ and $h(R) \propto R^{p+1}$ we have:

$$|1 + w_{\rm de}^{\rm eff}|\Omega_{\rm de}^{\rm eff} \approx 0.32|ph'(R)|\,|1 + 0.050(p-1)|\,.$$

The cosmological thin-shell constraint ensures that cosmologically $|h'(R)| \lesssim 10^{-4}$ today and so:

$$|1 + w_{\rm de}^{\rm eff}|\Omega_{\rm de}^{\rm eff} \lesssim 3.2|p|\,|1 + 0.050(p-1)| \times 10^{-5}.$$

3. Collected Constraints

We will consider now how the Eöt-Wash data, when thin-shells are assumed, constrains the properties of power-law $f(R)$ theories. It should be stressed that in the absence of thin-shells, the Eöt-Wash would automatically rule out these theories.

Defining $\bar{R} = \Lambda_0^4/M_{\rm Pl}^2$ we have plotted the Eöt-Wash constraints on Λ_0 for $-5 < p < 1$ in Fig. 1 as a thick (blue) solid line. The cosmological thin-shell constraint is shown as a thick (red) dashed line. For theories with $0 < p < 1$ we find a lower bound on Λ_0 and for theories with $p < 0$ we recover an upper bound. We also show, as a thick (black) dotted line, the naïve constraint on the parameters that one would find by simply requiring that the chameleon mass at the minimum of the effective inside the plate, m_c, is large compared to the plate thickness $D_p = 0.997\,{\rm mm}$. It is a commonplace assumption in the literature (see e.g. [34, 37]) that assuming $m_cD_{\rm p} \gg 1$ (where more generally $D_{\rm p}$ would be the length scale of the test body) is enough to satisfy local tests of gravity. It is clear from the plots that this naïve bound never provides the tightest constraint on the parameters of the theory.

The constraints on $\bar{R}$ constrain the equation of state parameter of the dark energy described by the $f(R)$ theory. Taking $\Omega_m = 0.23$ today, we plot the collected constraints on the effective Jordan frame equation of state parameter (as defined in Section III) in Fig. 2. We see that at the current epoch $|1 + w_{\rm de}| < 10^{-4}$ for all $\mathcal{O}(1)$ values of p with the largest values occurring for $p < -1$, and hence the late time cosmology produced by any viable theory would be observationally indistinguishable from that described by the standard ΛCDM model.

VI. CONCLUSIONS

In recent years, modifications of General Relativity have been suggested as a possible explanation for the observed accelerated expansion of the universe. A popular class of models are the so–called $f(R)$ theories. While cosmologically viable theories can be found, local constraints on such theories have to be worked out, since the gravitational sector is modified, which could result in unacceptable deviations from Newton's law of gravity.

In this paper we have constrained $f(R)$ theories, using the well known equivalence between these and scalar-tensor theories. For an $f(R)$ theory to be consistent with both cosmology and local gravity experiments, the equivalent scalar-tensor theory must be a chameleon field theory. We have shown that the requirement of the thin-shell mechanism at work in Eöt-Wash experiments results in an equation of state for dark energy very near to that of a cosmological constant. Thus, viable $f(R)$ models (those which are compatible with local experiments) behave on cosmological scales similarly to the standard ΛCDM model and deviations are expected only on very

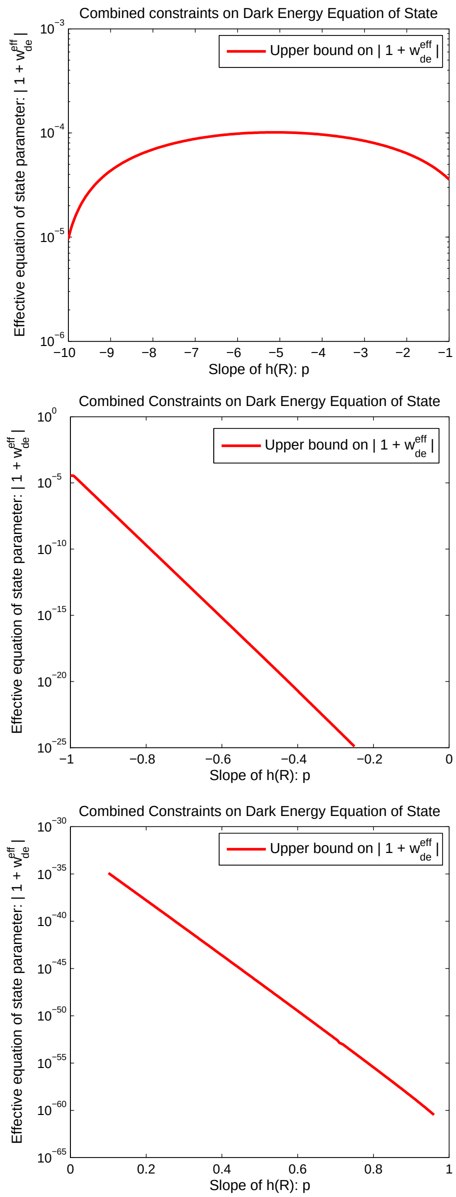

FIG. 2: Combined Eöt-Wash constraints on the effective dark energy equation of state parameter produced by $f(R)$ gravity theories with $f(R) = R + h(R)$ where $h(R) = \bar{R}(R/\bar{R})^{p+1}$. $\bar{R} = \Lambda_0^4/M_{\rm Pl}^2$ and $-5 < p < -1$, $-1 < p < 0$ and $0 < p < 1$. These constraints have been derived by requiring both that the Eöt-Wash test masses have thin-shells and by requiring that the chameleonic torque produced between the two thin-shelled test masses is small enough to have avoided detection to date. We see that in all cases we have $|1 + w_{\rm de}^{\rm eff}| < 10^{-4}$ today. As a result, the late time cosmology of any viable theory would be virtually indistinguishable from that described by the ΛCDM model

small (sub-galactic) scales. The expected deviations from the cosmological constant equation of state $w = -1$ now in viable $f(R)$ theories are unmeasurably small (at least with current technologies). As examples, we have studied $f(R)$ theories with logarithmic potentials (based on [37] for a fixed coupling $\beta = 1/\sqrt{6}$) as well as power-law potentials (such as those presented in [26, 34]). The former are ruled out by local gravitational tests, while there is still room for the latter models.

To conclude, while on cosmological scales viable $f(R)$ theories behave like ΛCDM, deviations are expected on scales which could be large enough to be within the reach of next generation galaxy surveys [25]. Hopefully, future measurements of the dark matter distribution on those scales can be used to find such deviations from the standard ΛCDM model. For this, a detailed understanding of galaxy formation is necessary, including an understanding of both the dynamics of baryons as well as that of dark matter in ΛCDM and $f(R)$/chameleon theories.

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APPENDIX A: THE FORCE BETWEEN TWO PLATES

In previous works [28, 31], the chameleonic force per unit area between two parallel plates was calculated and found to be:

$$\frac{F_\phi}{A} = V'(\phi_0) - V'(\phi_b) + \frac{\beta\rho_b}{M_{\rm Pl}}(\phi_0 - \phi_b),$$

where ϕ_0 depends on d. We assume that both plates have radius R and thickness D and that $D \ll R$. However as these calculations treat the plates as being infinite, they required for consistency that:

- *either*, for an isolated plate, $\phi \to \phi_b$ at a distance, $d \ll R$, from the plate so that the infinite plate approximation was still valid,

- *or*, the precise value of ϕ_b was not important when the plates were separated by a distance $d \ll R$. This means that provided $V'(\phi_0)/V'(\phi^*) \ll 1$, one could replace ϕ_b in the above expression by ϕ^* without altering the prediction for F_ϕ/A greatly. In these cases, the behaviour of ϕ far from the plates, where the infinite plate approximation is invalid, would be unimportant.

These approximations held for all of the chameleon theories considered in Refs. [28, 31], however in this work we consider a wider range of theories, and it is often the case that both of these assumptions fail to hold. In this appendix, we therefore derive an improved version of the force formula.

Outside of a body in a region where the background density is ρ_b, the chameleon field obeys:

$$\nabla^2\phi = V_{,\phi}(\phi) + \frac{\beta\rho_b}{M_{\rm Pl}}.$$

Consider this equation near one of the circular surfaces of a cylindrical plate, of uniform density, and with radius R and thickness $D \ll R$. Defining z to be the distance from one of the circular surfaces of the plate we have:

$$\frac{\partial^2\phi}{\partial z^2} = V_{,\phi}(\phi) - V_{,\phi}(\phi_b) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right). \tag{A1}$$

We begin by consider the case where only one plate is present. Here $\phi \to \phi_b$, $\partial\phi/\partial z \to 0$ as $z \to \infty$. Integrating Eq. (A1) with these boundary conditions give:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 = \; & V(\phi) - V(\phi_b) - V_{,\phi}(\phi_b)(\phi - \phi_b) \\ & + \int_z^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)\frac{\partial\phi}{\partial z}{\rm d}z. \end{aligned} \tag{A2}$$

We solve this approximately by assuming that for $z < z^*$ the z-dependence of the r-derivative terms is weak compared to that of the potential terms, and that for $z > z^*$, the non-linear terms in the potential, i.e. terms that depend on 3rd or higher derivatives of V, are subdominant. In $z > z^*$, we have $\phi \approx \tilde{\phi}$ where:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial\tilde{\phi}}{\partial z}\right)^2 = \; & \frac{1}{2}m_b^2(\tilde{\phi} - \phi_b)^2, \\ & + \int_z^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}}{\partial r}\right)\frac{\partial\tilde{\phi}}{\partial z}{\rm d}z. \end{aligned} \tag{A3}$$

or equivalently:

$$\frac{\partial^2\tilde{\phi}}{\partial z^2} \approx m_b^2(\tilde{\phi} - \phi_b) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}}{\partial r}\right).$$

Assuming that the plate is thin ($D \ll R$), and solving Eq. (A3), we find that for the $z > z^*$ and along $r = 0$:

$$\tilde{\phi} - \phi_b \propto \left(e^{-m_b z} - e^{-m_b r}\right),$$

and here z is the distance from the plate surface. It follows that for $z \ll R$ we have:

$$\int_z^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}}{\partial r}\right)\frac{\partial\tilde{\phi}}{\partial z}{\rm d}z = m_b^2(\tilde{\phi} - \phi_b)^2\frac{(e^{m_b R} - 1/2)}{(e^{m_b R} - 1)^2}. \tag{A4}$$

In $z < z^*$, we have assumed that the z-dependence of the r-gradient terms is relativity weak. We therefore approximate the r gradient terms in Eq. (A2) using the $z > z^*$ solution i.e. we approximate them using Eq. (A4) with $\tilde{\phi} \to \phi$. For $z \ll R$ we then have

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 \; & \approx V(\phi) - V(\phi_b) + V_{,\phi}(\phi_b)(\phi - \phi_b) \\ & + m_b^2(\phi - \phi_b)^2\frac{(e^{m_b R} - 1/2)}{(e^{m_b R} - 1)^2}. \end{aligned} \tag{A5}$$

The above equation also holds approximately in the $z > z^*$ region, provided $z \ll R$, and so provides an approximation to the evolution of ϕ everywhere when $z \ll R$. In particular we see that on the surface of the plate, at $z = 0$, where say $\phi = \bar{\phi}_s$:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 \; & \approx V(\bar{\phi}_s) - V(\phi_b) + V_{,\phi}(\phi_b)(\bar{\phi}_s - \phi_b) \\ & + m_b^2(\bar{\phi}_s - \phi_b)^2\frac{(e^{m_b R} - 1/2)}{(e^{m_b R} - 1)^2}. \end{aligned} \tag{A6}$$

We assume that the plate has a thin-shell, so that deep inside it $\phi \to \phi_c$ where:

$$V_{,\phi}(\phi_c) = -\frac{\beta\rho_c}{M_{\rm Pl}}.$$

Provided the shell is thin, we can treat the system as being essentially 1 dimensional [28, 31] and so:

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 = V(\phi) - V(\phi_c) - V_{,\phi}(\phi_c)(\phi - \phi_c). \tag{A7}$$

Thus by evaluating and equating the left hand sides of Eqs. (A6) and (A7) at the surface we find:

$$\begin{aligned} & V(\phi_c) - V(\phi_b) - V_{,\phi}(\phi_c)(\phi_c - \phi_b) + (V_{,\phi}(\phi_c) \\ & - V_{,\phi}(\phi_b))(\bar{\phi}_s - \phi_b) \\ & + m_b^2 C(m_b R)(\bar{\phi}_s - \phi_b)^2 = 0, \end{aligned}$$

where

$$C(m_b R) = \frac{(e^{m_b R} - 1/2)}{(e^{m_b R} - 1)^2}.$$

Thus:

$$\begin{aligned}\bar{\phi}_s - \phi_b \; & = \frac{(V_{,\phi}(\phi_b) - V_{,\phi}(\phi_c))}{2m_b^2 C(m_b R)}\Big[1 \\ & - \sqrt{1 + 4C(m_b R)D(\phi_c, \phi_b)}\Big] \end{aligned} \tag{A8}$$

where

$$D(\phi_c, \phi_b, m_b R) = \frac{m_b^2\left[V(\phi_b) - V(\phi_c) - V_{,\phi}(\phi_c)(\phi_b - \phi_c)\right]}{(V_{,\phi}(\phi_b) - V_{,\phi}(\phi_c))^2}.$$

We now consider the force between two parallel plates. This derivations make uses of results found in Refs. [28] and [31], and proceeds along roughly similar lines.

In between two parallel plates with radius R and with separation $d \ll R$ in the z-direction, the chameleon field obeys:

$$\frac{\partial^2\phi}{\partial z^2} = V_{,\phi}(\phi) - V_{,\phi}(\phi_b) \tag{A9}$$

For simplicity we treat the plates as having the same composition. This assumption was dropped in Ref. [28], however, it was also shown there that for most purposes the assumption provides an excellent approximation. This is because the chameleonic force generally exhibits very little composition dependence [28]. We define $z = 0$ to be the surface of one of the plates, and $z = d$ to be the facing surface of the second plate. The system is symmetric and so $\mathrm{d}\phi/\mathrm{d}z = 0$ at $z = d/2$. We define $\phi(z = d/2) = \phi_0(d)$. A formulae for $\phi_0(d)$ have been provided in Ref. [31]. Integrating Eq. (A9) we have:

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 = V(\phi) - V(\phi_0) - V_{,\phi}(\phi_b)(\phi - \phi_0). \quad \text{(A10)}$$

Following Ref. [31], inside either plate, Eq. (A7) holds. By equating both Eqs. (A10) and (A7) at the surface of one of the plates, where $\phi = \phi_s$ say, we find:

$$\phi_s = \frac{[V(\phi_c) - V(\phi_0) + V_{,\phi}(\phi_b)\phi_0 - V_{,\phi}(\phi_c)\phi_c]}{V_{,\phi}(\phi_b) - V_{,\phi}(\phi_c)}.$$

In Ref. [31] it was shown that the attractive chameleonic force unit area between two thin-shelled plates is given by $-V_{,\phi}(\phi_c)(\phi_s - \bar{\phi}_s)$, and if, as is usually the case, the plates are much denser than their environment so that $V_{,\phi}(\phi_c)/V_{,\phi}(\phi_b) = \rho_c/\rho_b \gg 1$, we have:

$$\frac{F_\phi}{A} = V(\phi_0) - V(\phi_b) - V_b'(\phi_0 - \phi_b) + m_b^2 C(m_b R)(\bar{\phi}_s - \phi_b)^2. \quad \text{(A11)}$$

This coincides with the formulae found in Refs. [28, 31] when $m_b R \gg 1 \Rightarrow C(m_b R) \approx 0$, or more generally whenever the final term is small compared with the other terms, which is when $C(m_b R)D(\phi_c, \phi_b) \ll 1$. $\bar{\phi}_s - \phi_b$ is given by Eq. (A8).

When $m_b R \ll 1$, we have $C(m_b R) \approx 1/2m_b^2 R^2 \gg 1$. If this is the case we also have $C(m_b R)D(\phi_c, \phi_b) \gg 1$:

$$m_b^2 C(m_b R)(\bar{\phi}_s - \phi_b)^2 \approx V(\phi_b) - V(\phi_c) - V_{,\phi}(\phi_c)(\phi_b - \phi_c),$$

and then in this limit Eq. (A11) becomes:

$$\frac{F_\phi}{A} = V(\phi_0) - V(\phi_c) - V_c'(\phi_b - \phi_c) - V_b'(\phi_0 - \phi_b).$$

APPENDIX B: CHAMELEON MASS BETWEEN TWO PLATES

In this appendix we generalize the calculation of the chameleon mass between two parallel plates, as performed in Refs. [28, 31], to include the wider range of chameleon theories considered here.

In between two parallel plates with, say, a circular cross section, in the $x - y$ plane, with radius r_{p}, and a separation, d, in the z-direction where $d \ll r_{\mathrm{p}}$, Eq. (6) for ϕ simplifies to be essentially one dimensional $\Box \to \frac{\mathrm{d}^2}{\mathrm{d}z^2}$:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} = V_{,\phi}(\phi) - V_{,\phi}(\phi_b), \quad \text{(B1)}$$

where ϕ_b is the background value of ϕ. We define ϕ_0 to be the value of ϕ when $\mathrm{d}\phi/\mathrm{d}z = 0$, which will occur midway between the two plates when $z = d/2$.

Thus integrating the ϕ equation we find:

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 = V(\phi) - V(\phi_0) - V_{,\phi}(\phi_b)(\phi - \phi_b).$$

In this work we have considered power law potentials where $V \propto \epsilon p(-\epsilon\phi/(p+1))^{\frac{p+1}{p}}$ and where $\epsilon = \mathrm{sgn}(p(p+1))$ and $p < 1$. For these potentials we have $m_\phi^2 = V_{,\phi\phi} = (p+1)/p^2 V(\phi)/\phi^2$. Thus, defining $Y = \phi/\phi_0$ and $m_0 = m_\phi(\phi_0)$ we have:

$$\frac{1}{2}\left(\frac{\mathrm{d}Y}{\mathrm{d}z}\right)^2 = \frac{p^2 m_0^2}{(p+1)}\left[Y^{\frac{p+1}{p}} - 1 - \frac{p+1}{p}\left(\frac{\phi_b}{\phi_0}\right)^{\frac{1}{p}}(Y-1)\frac{V(\phi_0)}{\phi_0}\right]. \quad \text{(B2)}$$

Now

$$R \propto \phi^{1/p},$$

and so defining $R_0 = R(\phi_0)$ we have $(\phi_b/\phi_0)^{1/p} = R_b/R_0$. Thus when $R_b/R_0 \ll 1$, the last term in Eq. (B2) is very small and can be dropped. Working in this limit we have:

$$\frac{1}{2}\left(\frac{\mathrm{d}Y}{\mathrm{d}z}\right)^2 \approx \frac{p^2 m_0^2}{(p+1)}\left[Y^{\frac{p+1}{p}} - 1\right].$$

Now on the surface of the plates $\phi \sim \mathcal{O}(\phi_c)$ where ϕ_c is the value of ϕ inside the body, and assuming $R_c \gg R_0$ i.e. $m_c \gg m_0$, we can we treat $(\phi/\phi_0)^p$ as becoming very large as $z \to 0$ (i.e. as we approach the surface of the plate). We define $X = Y^{-p} = (\phi/\phi_0)^{-p}$ and then in the limit $m_b \ll m_0 \ll m_c$ we have:

$$\left(\frac{\mathrm{d}X}{\mathrm{d}z}\right)^2 \approx \frac{2m_0^2 X^{2+2/p}}{(p+1)}\left[X^{-\frac{p+1}{p^2}} - 1\right].$$

Integrating this equation and using $X = 0$ at $z = 0$, $X = 1$ at $z = d/2$ we have for $p \le -1$:

$$\frac{m_0 d}{\sqrt{2|p+1|}} = \frac{p^2}{|p+1|} B\left(\frac{1}{2}, \frac{p)}{(1+p)}\right),$$

which simplifies to:

$$m_0 d = \sqrt{\frac{2}{|1+p|}} p^2 B\left(\frac{1}{2}, \frac{p}{(1+p)}\right). \quad \text{(B3)}$$

If $-1 \le p \le 1$ then we find:

$$m_0 d = \sqrt{\frac{2}{|1+p|}} B\left(\frac{1}{2}, \frac{1-p}{2(1+p)}\right). \quad \text{(B4)}$$

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